Let X be a discrete random variable with probability mass function

$$f_X(x) = \begin{cases} \dfrac{e^{-\frac{3}{4}}\left(\frac{3}{4}\right)^{x+1}}{x!} + c\left(\frac{3}{4}\right)^x, & x \in \{0,1,2,\cdots\} \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the value of c

(b) Calculate the value of probability generating function at $\frac{1}{2}$, $G_X(\frac{1}{2})$.

Solution:

(a)

$$\sum_{x=0}^{\infty} f_X(x) = \sum_{x=0}^{\infty}\left(\frac{e^{-\frac{3}{4}}\left(\frac{3}{4}\right)^{x+1}}{x!} + c\left(\frac{3}{4}\right)^x\right) = \frac{3}{4}e^{-\frac{3}{4}}\sum_{x=0}^{\infty}\frac{\left(\frac{3}{4}\right)^x}{x!}$$

$$+c\sum_{x=0}^{\infty}\left(\frac{3}{4}\right)^x = \frac{3}{4}e^{-\frac{3}{4}}e^{\frac{3}{4}} + c\cdot\frac{1}{1-\frac{3}{4}} = \frac{3}{4} + c\cdot 4 = 1$$

$$c = \frac{1}{16}$$

(b)

$$f_X(x) = \begin{cases} \dfrac{e^{-\frac{3}{4}}\left(\frac{3}{4}\right)^{x+1}}{x!} + \frac{1}{16}\left(\frac{3}{4}\right)^x & x \in \{0,1,2,\cdots\} \\ 0 & \text{otherwise} \end{cases}$$

$$G_X(t) = E(t^X) = \sum_{x=0}^{\infty} t^x f_X(x) = \sum_{x=0}^{\infty}\left(\frac{e^{-\frac{3}{4}}\left(\frac{3}{4}\right)^{x+1}}{x!}\cdot t^x + t^x\frac{1}{16}\left(\frac{3}{4}\right)^x\right)$$

$$= \frac{3}{4}e^{-\frac{3}{4}}\sum_{x=0}^{\infty}\frac{\left(\frac{3}{4}t\right)^x}{x!} + \frac{1}{16}\sum_{x=0}^{\infty}\left(\frac{3t}{4}\right)^x = \frac{3}{4}e^{-\frac{3}{4}}e^{\frac{3}{4}t} + \frac{1}{16}\cdot\frac{1}{1-\frac{3t}{4}}$$

$$= \frac{3}{4}e^{\frac{3}{4}(t-1)} + \frac{1}{4(4-3t)}, \quad \left(|t| < \frac{4}{3}\right)$$

$$G_X\left(\frac{1}{2}\right) = \frac{3}{4}e^{\frac{3}{4}\left(\frac{1}{2}-1\right)} + \frac{1}{4\left(4-3\cdot\frac{1}{2}\right)} = \frac{3}{4}e^{-\frac{3}{8}} - \frac{1}{10}$$