比较法、比值法、积分法，根值法等判别法例题

① $\sum_{k=3}^{\infty} \frac{1}{k^{\frac{5}{7}} \ln k}$ 与 $\sum_{k=3}^{\infty} \frac{1}{k \ln k}$ 极限比较

解：用积分法判定 $\sum_{k=3}^{\infty} \frac{1}{k \ln k}$，$\int_3^{\infty} \frac{1}{x \ln x} dx = \ln(\ln x) \Big|_3^{\infty} = \infty$ 发散.

$$\lim_{k\to\infty} \frac{\frac{1}{k^{5/7} \ln k}}{\frac{1}{k \ln k}} = \lim_{k\to\infty} \frac{k \ln k}{k^{5/7} \ln k} = \lim_{k\to\infty} \sqrt[7]{k^2} = \infty$$

由于 $\sum_{k=3}^{\infty} \frac{1}{k \ln k}$ 发散 所以 $\sum_{k=3}^{\infty} \frac{1}{k^{5/7} \ln k}$ 发散.

② $\sum_{k=1}^{\infty} \frac{7^k}{k^k}$

解：比值法：
$$\lim_{k\to\infty} \frac{\frac{7^{k+1}}{(k+1)^{k+1}}}{\frac{7^k}{k^k}} = \lim_{k\to\infty} \frac{k^k}{(k+1)^{k+1}} \cdot \frac{7^{k+1}}{7^k} = \lim_{k\to\infty} \frac{7}{k+1} \left(\frac{k}{k+1}\right)^k$$

$$= \lim_{k\to\infty} \left(1 + \frac{-1}{k+1}\right)^{\frac{k+1}{-1} \cdot \frac{-1}{k+1} \cdot k} \cdot \frac{7}{k+1} = \lim_{k\to\infty} \frac{7e^{-1}}{k+1} = 0 < 1$$

$\therefore \sum_{k=1}^{\infty} \frac{7^k}{k^k}$ 收敛.

③ $\sum_{k=1}^{\infty} \frac{16}{2^{\ln k} + 2}$

解：$e \approx 2.718 > 2$，$k = e^{\ln k} > 2^{\ln k}$

$\therefore \frac{16}{2^{\ln k} + 2} \geq \frac{16}{k+2}$，$\sum_{k=1}^{\infty} \frac{16}{k+2}$ 与 $\sum_{k=1}^{\infty} \frac{1}{k}$ 极限比较

$\lim_{k\to\infty} \frac{\frac{16}{k+2}}{\frac{1}{k}} = 16$. $\sum_{k=1}^{\infty} \frac{1}{k}$ 发散，$\sum_{k=1}^{\infty} \frac{16}{k+2}$ 发散，$\sum_{k=1}^{\infty} \frac{16}{2^{\ln k} + 2}$ 发散.