几种常用变换方法解一阶微分方程典型例题

(1) $y'=\frac{e^y+3x}{x^2}$

解：$\frac{dy}{dx}=\frac{e^y}{x^2}+\frac{3}{x}$. 令 $p=e^{-y}$

$\frac{dp}{dx}=-e^{-y}\frac{dy}{dx}$. $\frac{dy}{dx}=-\frac{1}{p}\frac{dp}{dx}$. 代入方程得

$-\frac{1}{p}\frac{dp}{dx}=\frac{1}{x^2}\frac{1}{p}+\frac{3}{x}$, $\frac{dp}{dx}+\frac{3}{x}p=-\frac{1}{x^2}$

$\mu(x)=e^{\int p(x)dx}=e^{\int\frac{3}{x}dx}=e^{3\ln x}=x^3$

$p=\frac{1}{x^3}\left[\int x^3\left(-\frac{1}{x^2}\right)dx+C\right]=\frac{1}{x^3}\left(-\frac{1}{2}x^2+C\right)$

$=-\frac{1}{2x}+\frac{C}{x^3}$. $e^{-y}=-\frac{1}{2x}+\frac{C}{x^3}$

$x^2(2xe^{-y}+1)=C$

(2) $y'^2+(x+y)y'+xy=0$

解：$y'^2+xy'+yy'+xy=0$

$y'(y'+x)+y(y'+x)=0$

$(y'+x)(y'+y)=0$. $y'=-x$ $y=-\frac{1}{2}x^2+C$

$y'+y=0$ $y=Ce^{-x}$.

(3) $y' = \dfrac{2x^3+3xy^2+x}{3x^2y+2y^3-y}$

解：$\begin{cases} 2x^3+3xy^2+x=0 \\ 3x^2y+2y^3-y=0 \end{cases} \Rightarrow \begin{cases} 2x^2+3y^2+1=0 \\ 3x^2+2y^2-1=0 \end{cases}$

$\begin{matrix} x^2=1 \\ y^2=-1 \end{matrix}$ $\quad \begin{matrix} u=x^2-1 \\ v=y^2+1 \end{matrix}$ $\quad \dfrac{dv}{du}=\dfrac{y}{x}\dfrac{dy}{dx}$

$\dfrac{dy}{dx}=\dfrac{x}{y}\dfrac{(2x^2+3y^2+1)}{(3x^2+2y^2-1)}$ $\quad \dfrac{y}{x}\dfrac{dy}{dx}=\dfrac{2x^2+3y^2+1}{3x^2+2y^2-1}$

$$\frac{dv}{du}=\frac{2(u+1)+3(v-1)+1}{3(u+1)+2(v-1)-1}=\frac{2u+3v}{3u+2v}=\frac{2+3\frac{v}{u}}{3+2\frac{v}{u}}$$

$p=\frac{v}{u}$，$v=pu$，$\dfrac{dv}{du}=p+u\dfrac{dp}{du}$

$p+u\dfrac{dp}{du}=\dfrac{2+3p}{3+2p}$，$u\dfrac{dp}{du}=\dfrac{2-2p^2}{3+2p}$

$\dfrac{3+2p}{1-p^2}dp=\dfrac{2}{u}du$，$\displaystyle\int\frac{3+2p}{1-p^2}dp=\int\frac{2}{u}du$

$\displaystyle\int\frac{5/2}{1-p}+\frac{\frac{1}{2}}{1+p}dp=\int\frac{2}{u}du$，$-\frac{5}{2}\ln|1-p|+\frac{1}{2}\ln|1+p|=2\ln|u|+\ln C$

$\sqrt{\dfrac{1+p}{(1+p)^5}}=Cu^2$. $p=\dfrac{v}{u}=\dfrac{y^2+1}{x^2-1}$. $u=x^2-1$

$$\frac{(x^2-y^2-2)^5}{x^2+y^2}=C$$