题：已知 $X_1, X_2, \cdots X_n$ 是取自正态总体 $N(\mu, \sigma^2)$ 的随机样本. $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $S^2 = \frac{1}{n-1}\sum(X_i-\bar{X})^2$

求证：(1) $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$ (2) $E(S^2) = \sigma^2$

证明：

$$\because \sum(X_i-\bar{X})^2 = \sum(X_i-\mu+\mu-\bar{X})^2$$

$$= \sum\left((X_i-\mu)^2 + 2(X_i-\mu)(\mu-\bar{X}) + (\mu-\bar{X})^2\right) = \sum(X_i-\mu)^2$$

$$-2\sum(X_i-\mu)(\bar{X}-\mu) + \sum(\bar{X}-\mu)^2 = \sum(X_i-\mu)^2$$

$$-2\sum(X_i-\bar{X}+\bar{X}-\mu)(\bar{X}-\mu) + n(\bar{X}-\mu)^2 = \sum(X_i-\mu)^2$$

$$-2\sum(X_i-\bar{X})(\bar{X}-\mu) - 2\sum(\bar{X}-\mu)^2 + n(\bar{X}-\mu)^2$$

$$= \sum(X_i-\mu)^2 - 0 - 2n(\bar{X}-\mu)^2 + n(\bar{X}-\mu)^2$$

$$= \sum(X_i-\mu)^2 - n(\bar{X}-\mu)^2$$

(1)
$$\frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{(n-1)\sigma^2}}}$$

$$= \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{\sum(X_i-\mu)^2 - n(\bar{X}-\mu)^2}{(n-1)\sigma^2}}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 - \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2}{n-1}}}$$

$$= \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{\chi_n^2-\chi_1^2}{n-1}}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\chi_{n-1}^2/n-1}} \sim t_{n-1}$$

(2)
$$E(S^2) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)$$

$$= \frac{1}{n-1}E\left(\sum(X_i-\mu)^2 - n(\bar{X}-\mu)^2\right) = \frac{1}{n-1}\left(\sum E(X_i-\mu)^2\right.$$

$$\left. - nE(\bar{X}-\mu)^2\right) = \frac{1}{n-1}\left(n\sigma^2 - n\cdot\frac{\sigma^2}{n}\right) = \sigma^2$$

注：

$\sum_{i=1}^{n}(\bar{X}-\mu) = n(\bar{X}-\mu)$

$\sum(X_i-\bar{X}) = 0$

$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$\frac{X_i-\mu}{\sigma} \sim N(0,1)$

$\frac{N(0,1)}{\sqrt{\chi_{n-1}^2/n-1}} \sim t_{n-1}$

$E(X_i-\mu)^2 = Var(X_i) = \sigma^2$

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$E(\bar{X}-\mu)^2 = var(\bar{X}) = \frac{\sigma^2}{n}$